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A THEOREM IN THE THEORY OF FINITE ELASTIC DEFORMATIONS

by

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ABSTRACT

In a previous paper, it has been shown that the displacements produced in a body of elastic material by a specified system of applied forces can be calculated according to second order elasticity theory by (i) calculating the displacements produced by the specified system of forces according to the first order (i.e. classical) elasticity theory, (ii) calculating the additional forces which must be applied to the body according to the second order theory in order to produce these displacements and (iii) calculating the displacements which are produced in the body according to classical elasticity theory by this additional set of forces. Then, the displacements produced in the body by the specified set of forces, according to second order elasticity theory, is given by subtracting the displacements calculated according to (iii) from those calculated according to (i).

In the previous paper, this theorem was proven for an isotropic elastic material. In the present paper, a proof of the theorem is given which is valid also for anisotropic materials. Furthermore, the theorem is extended to provide a method for calculating the displacements produced in a body of elastic material by a specified force system, according to nth order elasticity theory.

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1. Introduction. In the theory of the finite deformation of bodies of elastic material, the elastic properties of the material may be defined by means of a strain-energy function. This is the energy stored elastically per unit volume of the material measured in its undeformed state and may be denoted by W. If, in the deformation, a point of the body which is initially at X_i, in a rectangular Cartesian co-ordinate system x_i, moves to x_i in the same co-ordinate system, then W may be expressed as a function of the components of the tensor g_{ij} defined by

$$g_{ij} = \frac{\partial x_k}{\partial X_i} \frac{\partial x_k}{\partial X_j} , \tag{1}$$

provided that the body is homogeneous in its undeformed state. If the material of the body is isotropic in its undeformed state, then W may be expressed in terms of the quantities g_{ij} through the three scalar invariants I_1, I_2 and I_3 of the tensor g_{ij} defined by

$$I_1 = g_{ii} , \; I_2 = G_{ii} \text{ and } I_3 = \det g_{ij} , \tag{2}$$

where G_{ij} is the co-factor of $_{ij}$ in $\det g_{ij}$.

If the material, whether isotropic or not, obeys the equations of classical elasticity theory for sufficiently small deformations, then W must be expressible as a polynomial in g_{ij} and hence as a polynomial in $\partial u_i/\partial X_j$, where

$$u_i = x_i - X_i . \tag{3}$$

We may therefore write

$$W = \sum_{n=o}^{\infty} W_n , \tag{4}$$

where W_n is a homogeneous polynomial of degree n in the nine displacement gradients $\partial u_i/\partial X_j$. If we take $W = 0$ when the body is undeformed, i.e. when $\partial u_i/\partial X_j = 0$, we have $W_o = 0$. Since the stress components t_{ij} in the co-ordinate system x_i are given by

$$t_{ij} = \frac{1}{I_3^{\frac{1}{2}}} (\delta_{ik} + \frac{\partial u_i}{\partial X_k}) \frac{\partial W}{\partial(\partial u_j/\partial X_k)} , \tag{5}$$

if we assume that the stress in the body is zero in the undeformed state, we obtain $W_1 = 0$. We may thus write

$$W = \sum_{n=2}^{\infty} W_n . \tag{6}$$

If the deformation is such that $\partial u_i/\partial X_j$ is sufficiently small compared with unity, we can approximate to W by the expression W_2. To a higher degree of approximation we can take $W = W_2 + W_3$. To a still higher degree of approximation we can take $W = W_2 + W_3 + W_4$ and so on. Introducing these expressions for W into the expressions (5) for t_{ij} , we obtain corresponding expressions for the stress components.

The equations of motion and boundary conditions for the deformation of the body are given by

$$\frac{\partial}{\partial X_j} \left[\frac{\partial W}{\partial(\partial u_i/\partial X_j)} \right] + \rho_o f_i = \rho_o \frac{\partial^2 u_i}{\partial t^2} ,$$

$$\text{and } F_i = \frac{\partial W}{\partial(\partial u_i/\partial X_j)} \ell_j , \tag{7}$$

where f_i is the applied body force per unit mass, F_i is the applied surface traction per unit area of surface measured in the undeformed state of the body, ρ_0 is the density of the material in its undeformed state and ℓ_i are the direction-cosines of the normal to the surface in its undeformed state.

If we introduce $W = W_2$ into the equations (7), we obtain

$$\frac{\partial}{\partial X_j}\left[\frac{\partial W_2}{\partial(\partial u_i/\partial X_j)}\right] + \rho_0 f_i = \rho_0 \frac{\partial^2 u_i}{\partial t^2}$$

$$\text{and } F_i = \frac{\partial W_2}{\partial(\partial u_i/\partial X_j)} \ell_j \, . \tag{8}$$

These are, of course, the equations of motion and boundary conditions of classical elasticity theory. If f_i and F_i are specified and $\partial u_i/\partial X_j$ are sufficiently small compared with unity, the solutions for u_i of these equations provide a first approximation to the displacements produced in the body by the applied forces.

If we introduce $W = W_2 + W_3$ into the equations (7), we obtain

$$\frac{\partial}{\partial X_j}\left[\frac{\partial W_2}{\partial(\partial u_i/\partial X_j)}\right] + \frac{\partial}{\partial X_j}\left[\frac{\partial W_3}{\partial(\partial u_i/\partial X_j)}\right] + \rho_0 f_i = \rho_0 \frac{\partial^2 u_i}{\partial t^2}$$

$$\text{and} \quad F_i = \left[\frac{\partial W_2}{\partial(\partial u_i/\partial X_j)} + \frac{\partial W_3}{\partial(\partial u_i/\partial X_j)}\right] \ell_j \, . \tag{9}$$

These equations may be called the equations of motion and boundary conditions of second order elasticity theory. It will be shown that a solution for u_i of these equations may be obtained by the following procedure:

(i) we first obtain a solution $u_i = \varepsilon u_i' = \varepsilon u_i'(X_j)$ of the classical equations of motion and boundary conditions (8), valid to an order of approximation involving neglect only of terms of higher degree than the first in the space derivatives of the displacement components;

(ii) we introduce $u_i = \varepsilon u_i'$ into the equations (9) and thus calculate the applied body forces $f_i = f_i'$ (say) and surface tractions $F_i = F_i'$ (say) corresponding to the displacements $\varepsilon u_i'$ according to the equations of motion and boundary conditions of second order elasticity theory;

(iii) we now calculate, according to first order elasticity theory, the displacements $\varepsilon^2 u_i''$ which are produced in the body by the system of body forces $f_i' - f_i$ and surface tractions $F_i' - F_i$;

(iv) then $u_i = \varepsilon u_i' - \varepsilon^2 u_i''$ satisfies the equations (9) of second order elasticity theory with the neglect only of terms of higher degree than the second in the space derivatives of the displacement components.

This theorem has already been proven* for an isotropic material. The method of proof employed in the present paper (§2) is valid for both isotropic and anisotropic materials.

If we introduce $W = \sum_{r=2}^{n+2} W_r$ into the equations (7),

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* R.S. Rivlin "The Solution of Problems in Second Order Elasticity Theory". J. Rat'l Mech. & Anal. 2, 53-81 (1953).

we obtain

$$\sum_{r=2}^{n+2} \frac{\partial}{\partial X_j} \left[\frac{\partial W_r}{\partial(\partial u_i/\partial X_j)}\right] + \rho_o f_i = \rho_o \frac{\partial^2 u_i}{\partial t^2}$$

and

$$F_i = \sum_{r=2}^{n+2} \ell_j \frac{\partial W_r}{\partial(\partial u_i/\partial X_j)} \cdot \qquad (10)$$

These equations may be called the equations of motion and boundary conditions of (n+1)th. order elasticity theory.

It will be shown in § 3 that if, corresponding to an applied system of body forces f_i and surface tractions F_i, we have a displacement field $u_i = \varepsilon U_i^{(n)}$ satisfying the equations of nth. order elasticity theory, with the neglect only of terms of higher degree than the nth in the space derivatives of the displacement components, then we can find a displacement field $u_i = \varepsilon U_i^{(n+1)}$ satisfying the equations of (n+1)th order elasticity theory, for the same system of applied forces, with the neglect only of terms of higher degree than the (n+1)th in the space derivatives of the displacement components, by the following procedure:

(i) we introduce $u_i = \varepsilon U_i^{(n)}$ into the equations (10) of (n+1)th order elasticity theory and calculate the body forces f'_i (say) and surface tractions F'_i (say) corresponding to the displacements $\varepsilon U_i^{(n)}$ according to the equations of motion and boundary conditions of (n+1)th order elasticity theory;

(ii) we now calculate, according to first order elasticity theory, the displacements $\varepsilon^{n+1} u_i^{(n+1)}$ which are produced in the body by the system of body forces $f'_i - f_i$ and surface tractions $F_i' - F_i$;

(iii) then $u_i = \varepsilon U_i^{(n)} - \varepsilon^{n+1} u_i^{(n+1)}$ satisfies the equations of motion and boundary conditions of (n+1)th order elasticity theory, with the neglect only of terms of higher degree than the (n+1)th in the space derivatives of the displacement components.

It is apparent that by repetition of this process with n successively equal to 1,2,3,...n-1, we can calculate by this method the displacement field $\varepsilon U_i^{(n)}$ corresponding to the equations of nth order elasticity theory.

2. Second Order Approximations in Finite Elasticity Theory.

Let $u_i = \varepsilon u_i'$ be a solution of the equations (8), valid with the neglect only of terms of higher degree than the first in ε^2. Then,

$$\left[\frac{\partial}{\partial X_j} \left\{ \frac{\partial W_2}{\partial(\partial u_i/\partial X_j)} \right\} \right]_{u_i = \varepsilon u_i'} + \rho_0 (f_i + \varphi_i) = \varepsilon \rho_0 \frac{\partial^2 u_i'}{\partial t^2}$$

and
$$F_i + \Phi_i = \ell_j \left[\frac{\partial W_2}{\partial(\partial u_i/\partial X_j)} \right]_{u_i = \varepsilon u_i'} , \qquad (11)$$

where φ_i and Φ_i are $O(\varepsilon^2)$.

If we introduce $u_i = \varepsilon u_i'$ into equations (9), we find the expressions f_i' and F_i' for the forces which must be applied, according to second order elasticity theory, in order to maintain in the body concerned the deformation $u_i = \varepsilon u_i'$.

We have

$$\rho_o f_i' = \varepsilon\rho_o \frac{\partial^2 u_i'}{\partial t^2} - \left[\frac{\partial}{\partial X_j}\left\{\frac{\partial W_2}{\partial(\partial u_i/\partial X_j)}\right\} + \frac{\partial}{\partial X_j}\left\{\frac{\partial W_3}{\partial(\partial u_i/\partial X_j)}\right\}\right]_{u_i=\varepsilon u_i'}$$

and $$F_i' = \ell_j \left[\frac{\partial W_2}{\partial(\partial u_i/\partial X_j)} + \frac{\partial W_3}{\partial(\partial u_i/\partial X_j)}\right]_{u_i=\varepsilon u_i'} . \qquad (12)$$

From equations (11) and (12), we have

$$\rho_o(f_i' - f_i - \varphi_i) = -\left[\frac{\partial}{\partial X_j}\left\{\frac{\partial W_3}{\partial(\partial u_i/\partial X_j)}\right\}\right]_{u_i=\varepsilon u_i'}$$

and $$F_i' - F_i - \Phi_i = \ell_j \left[\frac{\partial W_3}{\partial(\partial u_i/\partial X_j)}\right]_{u_i=\varepsilon u_i'} . \qquad (13)$$

We see that $\rho_o(f_i' - f_i - \varphi_i)$ and $F_i' - F_i - \Phi_i$ are of second degree in ε^2.

Now, let us suppose that $\varepsilon^2 u_i''$ are the solutions of the first order (i.e. classical) equations of motion and boundary conditions (8), when the body forces and surface tractions are $f_i' - f_i$ and $F_i' - F_i$ respectively. We then have

$$\left[\frac{\partial}{\partial X_j}\left\{\frac{\partial W_2}{\partial(\partial u_i/\partial X_j)}\right\}\right]_{u_i=\varepsilon^2 u_i''} + \rho_o(f_i' - f_i) = \varepsilon^2\rho_o \frac{\partial^2 u_i''}{\partial t^2}$$

and $$F_i' - F_i = \ell_j \left[\frac{\partial W_2}{\partial(\partial u_i/\partial X_j)}\right]_{u_i=\varepsilon^2 u_i''} . \qquad (14)$$

We can show that $u_i = \varepsilon u_i' - \varepsilon^2 u_i''$ satisfies the equations of motion and boundary conditions of second order elasticity theory, with the neglect only of terms of higher degree than the second in ε .

Intoducing $u_i = \varepsilon u_i' - \varepsilon^2 u_i''$ into the equations (9), we have

$$\left[\frac{\partial}{\partial X_j}\left\{\frac{\partial W_2}{\partial(\partial u_i/\partial X_j)}\right\}\right]_{u_i = \varepsilon u_i' - \varepsilon^2 u_i''}$$

$$+\left[\frac{\partial}{\partial X_j}\left\{\frac{\partial W_3}{\partial(\partial u_i/\partial X_j)}\right\}\right]_{u_i = \varepsilon u_i' - \varepsilon^2 u_i''} + \rho_o f_i = \varepsilon\rho_o \frac{\partial^2 u_i'}{\partial t^2} - \varepsilon^2 \rho_o \frac{\partial^2 u_i''}{\partial t^2}$$

and
$$F_i = \ell_j \left[\frac{\partial W_2}{\partial(\partial u_i/\partial X_j)}\right]_{u_i = \varepsilon u_i' - \varepsilon^2 u_i''} + \ell_j \left[\frac{\partial W_3}{\partial(\partial u_i/\partial X_j)}\right]_{u_i = \varepsilon u_i' - \varepsilon^2 u_i''} . \tag{15}$$

Noting that $\partial W_3/\partial(\partial u_i/\partial X_j)$ is homogeneous and of second degree in the quantities $\partial u_p/\partial X_q$, we may replace

$$\left[\frac{\partial}{\partial X_j}\left\{\frac{\partial W_3}{\partial(\partial u_i/\partial X_j)}\right\}\right]_{u_i = \varepsilon u_i' - \varepsilon^2 u_i''} \text{ and } \left[\frac{\partial W_3}{\partial(\partial u_i/\partial X_j)}\right]_{u_i=\varepsilon u_i'-\varepsilon^2 u_i''}$$

by
$$\left[\frac{\partial}{\partial X_j}\left\{\frac{\partial W_3}{\partial(\partial u_i/\partial X_j)}\right\}\right]_{u_i=\varepsilon u_i'} \text{ and } \left[\frac{\partial W_3}{\partial(\partial u_i/\partial X_j)}\right]_{u_i=\varepsilon u_i'}$$

respectively, with the neglect only of terms of higher degree than the second in ε. Also, since $\partial W_2/\partial(\partial u_i/\partial X_j)$ is linear in $\partial u_p/\partial X_q$, we may write

$$\left[\frac{\partial}{\partial X_j}\left\{\frac{\partial W_2}{\partial(\partial u_i/\partial X_j)}\right\}\right]_{u_i=\varepsilon u_i'-\varepsilon^2 u_i''} = \left[\frac{\partial}{\partial X_j}\left\{\frac{\partial W_2}{\partial(\partial u_i/\partial X_j)}\right\}\right]_{u_i=\varepsilon u_i'}$$

$$- \left[\frac{\partial}{\partial X_j}\left\{\frac{\partial W_2}{\partial(\partial u_i/\partial X_j)}\right\}\right]_{u_i=\varepsilon^2 u_i''}$$

and $$\left[\frac{\partial W_2}{\partial(\partial u_i/\partial X_j)}\right]_{u_i=\varepsilon u_i'-\varepsilon^2 u_i''} = \left[\frac{\partial W_2}{\partial(\partial u_i/\partial X_j)}\right]_{u_i=\varepsilon u_i'}$$

$$- \left[\frac{\partial W_2}{\partial(\partial u_i/\partial X_j)}\right]_{u_i=\varepsilon^2 u_i''} . \qquad (16)$$

We may therefore re-write the equations (15) as

$$\left[\frac{\partial}{\partial X_j}\left\{\frac{\partial W_2}{\partial(\partial u_i/\partial X_j)}\right\}\right]_{u_i=\varepsilon u_i'} - \left[\frac{\partial}{\partial X_j}\left\{\frac{\partial W_2}{\partial(\partial u_i/\partial X_j)}\right\}\right]_{u_i=\varepsilon^2 u_i''}$$

$$+ \left[\frac{\partial}{\partial X_j}\left\{\frac{\partial W_3}{\partial(\partial u_i/\partial X_j)}\right\}\right]_{u_i=\varepsilon u_i'} + \rho_o f_i$$

$$= \varepsilon\rho_o \frac{\partial^2 u_i'}{\partial t^2} - \varepsilon^2 \rho_o \frac{\partial^2 u_i''}{\partial t^2}$$

and $$F_i = \ell_j \left[\frac{\partial W_2}{\partial(\partial u_i/\partial X_j)}\right]_{u_i=\varepsilon u_i'} - \ell_j \left[\frac{\partial W_2}{\partial(\partial u_i/\partial X_j)}\right]_{u_i=\varepsilon^2 u_i''}$$

$$+ \ell_j \left[\frac{\partial W_3}{\partial(\partial u_i/\partial X_j)}\right]_{u_i=\varepsilon u_i'} , \qquad (17)$$

with the neglect only of terms of higher degree than the second in ε.

Since u_i'' satisfies the equations (14) and u_i' satisfies the equations (11) and f_i' and F_i' are given by (12), we readily see that equations (17) are satisfied. Thus $u_i = \varepsilon u_i' + \varepsilon^2 u_i''$ satisfies the equations of motion and boundary conditions of second order elasticity theory, in which the body forces are f_i and the surface tractions F_i, with the neglect only of terms in the equations of higher degree than the second in ε.

Consequently, the displacement field which satisfies the equations of second-order elasticity theory, with the neglect only of terms of higher degree than the second in the space derivatives of the displacement components, may be calculated by the procedure described in §1.

3. Third and Higher Order Approximations in Finite Elasticity Theory.

Let us suppose that $u_i = \varepsilon U_i^{(n)} = \varepsilon u_i^{(1)} + \sum_{r=2}^{n} \varepsilon^r u_i^{(r)}$

satisfies the equations of motion and boundary conditions of nth order elasticity theory, with the neglect only of terms of higher degree than the nth in ε, when the body forces and surface tractions applied to the body under consideration are f_i and F_i respectively. We then have

$$\left[\sum_{r=2}^{n+1} \frac{\partial}{\partial X_j}\left\{\frac{\partial W_r}{\partial(\partial u_i/\partial X_j)}\right\}\right]_{u_i=\varepsilon U_i^{(n)}} + \rho_o (f_i + \varphi_i)$$
$$= \rho_o \frac{\partial^2 u_i^{(1)}}{\partial t^2} + \rho_o \sum_{r=2}^{n} \varepsilon^r \frac{\partial^2 u_i^{(r)}}{\partial t^2}$$

and

$$F_i + \Phi_i = \left[\sum_{r=2}^{n+1} \ell_j \frac{\partial W_r}{\partial(\partial u_i/\partial X_j)}\right]_{u_i=\varepsilon U_i^{(n)}} , \qquad (18)$$

where φ_i and Φ_i are of degree higher than n in ε. The body forces f_i' and surface tractions F_i' which must be applied to the body in order to support the deformation $u_i = \varepsilon u_i^{(1)} - \sum_{r=2}^{n} \varepsilon^r u_i^{(r)}$ according to (n+1)th order elasticity theory are given by

$$\left[\sum_{r=2}^{n+2} \frac{\partial}{\partial X_j}\left\{\frac{\partial W_r}{\partial(\partial u_i/\partial X_j)}\right\}\right]_{u_i=\varepsilon U_i^{(n)}} + \rho_o f_i'$$

$$= \rho_o \varepsilon \frac{\partial^2 u_i^{(1)}}{\partial t^2} - \rho_o \sum_{r=2}^{n} \varepsilon^r \frac{\partial^2 u_i^{(r)}}{\partial t^2}$$

and

$$F_i' = \left[\sum_{r=2}^{n+2} \ell_j \frac{\partial W_r}{\partial(\partial u_i/\partial X_j)}\right]_{u_i=\varepsilon U_i^{(n)}} . \tag{19}$$

From equations (18) and (19), we obtain

$$\left[\frac{\partial}{\partial X_j}\left\{\frac{\partial W_{n+2}}{\partial(\partial u_i/\partial X_j)}\right\}\right]_{u_i=\varepsilon U_i^{(n)}} + \rho_o (f_i' - f_i - \varphi_i) = 0$$

and

$$F_i' - F_i - \Phi_i = \ell_j \left[\frac{\partial W_{n+2}}{\partial(\partial u_i/\partial X_j)}\right]_{u_i=\varepsilon U_i^{(n)}} . \tag{20}$$

From these equations, we readily see that the expressions for $f_i' - f_i - \varphi_i$ and $F_i' - F_i - \Phi_i$ involve terms of degree n+1 or higher in ε. Since φ_i and Φ_i also involve terms of degree n+1 or higher in ε, we see that $f_i' - f_i$ and $F_i' - F_i$ involve terms of degree n+1 or higher in ε.

With the neglect only of terms of higher degree than n+1 in ε, equations (20) can be re-written as

$$\left[\frac{\partial}{\partial X_j}\left\{\frac{\partial W_{n+2}}{\partial(\partial u_i/\partial X_j)}\right\}\right]_{u_i=\varepsilon u_i^{(1)}} + \rho_o\,(f_i' - f_i - \varphi_i) = 0$$

and

$$F_i' - F_i - \Phi_i = \ell_j\left[\frac{\partial W_{n+2}}{\partial(\partial u_i/\partial X_j)}\right]_{u_i=\varepsilon u_i^{(1)}} . \qquad (21)$$

Now, let $\varepsilon^{n+1} u_i^{(n+1)}$ be the displacement produced in the body by the system of body forces $f_i' - f_i$ and surface tractions $F_i' - F_i$ according to the classical elasticity theory. We then have, from equations (8),

$$\left[\frac{\partial}{\partial X_j}\left\{\frac{\partial W_2}{\partial(\partial u_i/\partial X_j)}\right\}\right]_{u_i=\varepsilon^{n+1}u_i^{(n+1)}} + \rho_o\,(f_i' - f_i) = \rho_o\varepsilon^{n+1}\frac{\partial^2 u_i^{(n+1)}}{\partial t^2}$$

and

$$F_i' - F_i = \ell_j\left[\frac{\partial W_2}{\partial(\partial u_i/\partial X_j)}\right]_{u_i=\varepsilon^{n+1}u_i^{(n+1)}} . \qquad (22)$$

We can readily show that $u_i = \varepsilon u_i^{(1)} - \sum_{r=2}^{n+1} \varepsilon^r u_i^{(r)}$ satisfies the equations of motion and boundary conditions for (n+1)th order elasticity theory, for the problem in which the system of body forces f_i and surface tractions F_i are applied to the body under consideration, with the neglect only of terms of higher degree than n+1 in ε. If this is to be the case, we must have

$$\left[\sum_{r=2}^{n+2} \frac{\partial}{\partial X_j}\left\{\frac{\partial W_r}{\partial(\partial u_i/\partial X_j)}\right\}\right]_{u_i=\varepsilon U_i(n+1)} + \rho_o f_i$$

$$= \rho_o \varepsilon \frac{\partial^2 u_i(1)}{\partial t^2} - \rho_o \sum_{r=2}^{n+1} \varepsilon^r \frac{\partial^2 u_i(r)}{\partial t^2}$$

and

$$F_i = \ell_j \left[\sum_{r=2}^{n+2} \frac{\partial W_r}{\partial(\partial u_i/\partial X_j)}\right]_{u_i=\varepsilon U_i(n+1)}, \tag{23}$$

with the neglect of terms of higher degree than n+1 in ε.

To this degree of approximation, these equations may be re-written as

$$\left[\sum_{r=2}^{n+1} \frac{\partial}{\partial X_j}\left\{\frac{\partial W_r}{\partial(\partial u_i/\partial X_j)}\right\}\right]_{u_i=\varepsilon U_i(n)} + \left[\frac{\partial}{\partial X_j}\left\{\frac{\partial W_{n+2}}{\partial(\partial u_i/\partial X_j)}\right\}\right]_{u_i=\varepsilon u_i(1)}$$

$$- \left[\frac{\partial}{\partial X_j}\left\{\frac{\partial W_2}{\partial(\partial u_i/\partial X_j)}\right\}\right]_{u_i=\varepsilon^{n+1} u_i(n+1)} + \rho_o f_i$$

$$= \rho_o \varepsilon \frac{\partial^2 u_i(1)}{\partial t^2} - \rho_o \sum_{r=2}^{n+1} \varepsilon^r \frac{\partial^2 u_i(r)}{\partial t^2}$$

and

$$F_i = \ell_j \left[\sum_{r=2}^{n+1} \frac{\partial W_r}{\partial(\partial u_i/\partial X_j)}\right]_{u_i=\varepsilon U_i(r)}$$

$$+ \ell_j \left[\frac{\partial W_{n+2}}{\partial(\partial u_i/\partial X_j)}\right]_{u_i=\varepsilon u_i(1)}$$

$$- \ell_j \left[\frac{\partial W_2}{\partial(\partial u_i/\partial X_j)}\right]_{u_i=\varepsilon^{n+1} u_i(n+1)} . \tag{24}$$

From equations (18), (21) and (22), it follows that equations (24) are satisfied.

It is thus seen that if, corresponding to an applied system of body forces f_i and surface tractions F_i, we have a displacement field $u_i = \varepsilon U_i^{(n)}$ satisfying the equations of nth. order elasticity theory, with the neglect only of terms of higher degree than the nth. in the space derivatives of the displacement components, then we can find a displacement field $u_i = \varepsilon U_i^{(n+1)}$ satisfying the equations of (n+1)th. order elasticity theory, for the same system of applied forces, with the neglect only of terms of higher degree than the (n+1)th. in the displacement gradients, by the procedure described in §1. Consequently, by repeating the procedure described in §1 with n successively equal to 1,2,3,...n-1, we can calculate the displacement field $\varepsilon U_i^{(n)}$ corresponding to the equations of nth. order elasticity theory.

It will be noted that this procedure for obtaining a solution of the equations of nth. order elasticity theory is one which yields a unique solution. Other solutions of the equations may exist.